

# GAUSS SUMS OVER SOME MATRIX GROUPS

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**ABSTRACT.** In this note, we give explicit expressions of Gauss sums for general (resp. special) linear groups over finite fields, which involves Gauss sums (resp. Kloosterman sums). The key ingredient is averaging such sums over Borel subgroups. As applications, we count the number of invertible matrices of zero-trace over finite fields and we also improve two bounds by Ferguson, Hoffman, Luca, Ostafe and Shparlinski in [Some additive combinatorics problems in matrix rings, Rev. Mat. Complut. (23) 2010, 501–513].

## 1. INTRODUCTION

The Gauss sums for classical groups over a finite field have been extensively studied by Kim in several articles [8, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], and more recently, he also applied the Gauss sum for special linear groups over finite fields to coding theory [19].

Let  $q$  be a power of a prime number  $p$  and  $k = \mathbb{F}_q$  be the finite field with  $q$  elements. Let  $\lambda$  be a fixed nonprincipal additive character of  $\mathbb{F}_q$ , e.g, take

$$\lambda(x) = \exp\left(\frac{2\pi i}{p} \text{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)\right), \quad \forall x \in \mathbb{F}_q,$$

and  $\chi$  be a multiplicative character of  $\mathbb{F}_q^*$ .

Given two matrices  $U = (u_{ij}), V = (v_{ij}) \in M_n(k)$ , their product is defined by

$$(1.1) \quad U \cdot V = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} u_{ij} v_{ij}.$$

For  $U$  being a nonzero matrix of  $M_n(k)$ , let

$$(1.2) \quad G_{\text{GL}_n(k)}(U, \chi, \lambda) = \sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(U \cdot X),$$

and

$$(1.3) \quad G_{\text{SL}_n(k)}(U, \lambda) = \sum_{X \in \text{SL}_n(\mathbb{F}_q)} \lambda(U \cdot X).$$

These sums can be viewed as the general linear group and special linear group analogues of classical Gauss sums. For brevity, if  $\chi = 1$  is trivial, we will write  $G_{\text{GL}_n(k)}(U, \lambda)$  instead of  $G_{\text{GL}_n(k)}(U, 1, \lambda)$ .

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2000 *Mathematics Subject Classification.* 11C20, 11T23.

*Key words and phrases.* General linear group; Special linear group; Finite fields; Gauss sum; Kloosterman sum.

Kim [8] got the formulae of  $G_{\mathrm{GL}_n(k)}(I, \chi, \lambda)$  and  $G_{\mathrm{SL}_n(k)}(I, \lambda)$  by using the Bruhat decomposition of  $\mathrm{GL}_n(k)$  and  $\mathrm{SL}_n(k)$ . As Kim remarked, these formulae already appeared in the work of Eichler [1] and Lamprecht [22]. (See the introduction of [8]). Fulman [3] also got the same result for  $G_{\mathrm{GL}_n(k)}(I, \chi, \lambda)$  by using the technique of generating functions.

In this note, by using orthogonality of characters of finite abelian groups, we present an explicit expressions for the sums (1.2) and (1.3). The main idea in our approach is averaging the sums (1.2) and (1.3) over Borel subgroups, i.e, the group of upper triangular matrices. (See Theorems 2.1 and 2.4 below). As a consequence, we give the upper bounds of the sums (1.2) and (1.3) (in the case  $\chi = 1$ ). (See Corollaries 2.2 and 2.5 below).

By using several results from algebraic geometry, in particular, Skorobogatov's estimates of character sums along algebraic varieties [23], Ferguson, Hoffman, Luca, Ostafe and Shparlinski [2] also provided another upper bounds of the sums (1.2) and (1.3) and used them to study some additive combinatorics problems in matrix rings. Their bounds have been applied by us to study some uniform distribution properties of some matrix groups. (See [4]). Our bounds given in this note improve their bounds. (See Remarks 2.3 and 2.6 below).

Finally, as an application, we count the number of invertible matrices of zero-trace over finite fields.

## 2. MAIN RESULTS

**2.1. The case of  $\mathrm{GL}_n(k)$ .** The equation (1.2) can be rewritten as

$$(2.1) \quad G_{\mathrm{GL}_n(k)}(U, \chi, \lambda) = \sum_{X \in \mathrm{GL}_n(k)} \chi(\det X) \lambda(\mathrm{tr} U^t X),$$

where  $U^t$  is the transpose of  $U$  and “tr” stands for the trace of the matrix.

Replacing  $U$  by  $PUQ$  in (2.1) with  $P, Q \in \mathrm{GL}_n(k)$ , we get

$$\begin{aligned} & G_{\mathrm{GL}_n(k)}(PUQ, \chi, \lambda) \\ &= \sum_{X \in \mathrm{GL}_n(k)} \chi(\det X) \lambda(\mathrm{tr} Q^t U^t P^t X) \\ (2.2) \quad &= \sum_{X \in \mathrm{GL}_n(k)} \chi(\det X) \lambda(\mathrm{tr} U^t P^t X Q^t) \\ &= \bar{\chi}(\det PQ) \sum_{X \in \mathrm{GL}_n(k)} \chi(\det P^t X Q^t) \lambda(\mathrm{tr} U^t P^t X Q^t) \\ &= \bar{\chi}(\det PQ) G_{\mathrm{GL}_n(k)}(U, \chi, \lambda). \end{aligned}$$

Therefore,

$$(2.3) \quad G_{\mathrm{GL}_n(k)}(U, \chi, \lambda) = \chi(\det PQ) G_{\mathrm{GL}_n(k)}(PUQ, \chi, \lambda).$$

Let  $u$  be the rank of  $U$ . There exist  $P, Q \in \mathrm{GL}_n(k)$  such that

$$(2.4) \quad PUQ = \begin{pmatrix} I_u & 0 \\ 0 & 0 \end{pmatrix},$$

where  $I_u$  is the  $u \times u$  identity matrix. If  $u < n$ , additionally, we can also require  $P, Q \in \text{SL}_n(k)$ .

Combining equations (2.3) and (2.4), we get

$$(2.5) \quad G_{\text{GL}_n(k)}(U, \chi, \lambda) = \begin{cases} \bar{\chi}(\det U) \sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X) & \text{if } u = n \\ \sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X) & \text{if } u < n \end{cases},$$

where

$$(2.6) \quad \text{tr}_u X = \sum_{i=1}^u x_{ii}, \text{ for } X = (x_{ij}) \in \text{M}_n(k).$$

So it suffices to calculating the sum

$$(2.7) \quad \sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X), \text{ for } 1 \leq u \leq n.$$

Let  $\text{B}_n(k)$  be the Borel subgroup of  $\text{GL}_n(k)$ , i.e., the group of upper triangular invertible matrices. The following is the key step of our approach.

$$(2.8) \quad \begin{aligned} & \sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X) \\ &= \frac{1}{(q-1)^n q^{\binom{n}{2}}} \sum_{B \in \text{B}_n(k)} \sum_{X \in \text{GL}_n(k)} \chi(\det BX) \lambda(\text{tr}_u BX) \\ &= \sum_{X \in \text{GL}_n(k)} \frac{\chi(\det X)}{(q-1)^n q^{\binom{n}{2}}} \sum_{B \in \text{B}_n(k)} \chi(\det B) \lambda(\text{tr}_u BX) \\ &= \sum_{(x_{ij}) \in \text{GL}_n(k)} \frac{\chi(\det(x_{ij}))}{(q-1)^n q^{\binom{n}{2}}} \sum_{(b_{ij}) \in \text{B}_n(k)} \chi\left(\prod_{i=1}^n b_{ii}\right) \lambda\left(\sum_{\substack{i \leq j \\ i \leq u}} b_{ij} x_{ji}\right) \\ &= \sum_{(x_{ij}) \in \text{GL}_n(k)} \frac{\chi(\det(x_{ij}))}{(q-1)^n q^{\binom{n}{2}}} \prod_{i=1}^u \sum_{b_{ii} \neq 0} \chi(b_{ii}) \lambda(b_{ii} x_{ii}) \cdot \prod_{i=u+1}^n \sum_{b_{ii} \neq 0} \chi(b_{ii}) \\ & \quad \cdot \prod_{\substack{i < j \\ i \leq u}} \sum_{b_{ij} \in k} \lambda(b_{ij} x_{ji}) \cdot \prod_{\substack{i < j \\ i > u}} \sum_{b_{ij} \in k} 1. \end{aligned}$$

If  $\chi$  is not principal and  $u < n$ , then we have

$$(2.9) \quad \sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X) = 0$$

because

$$\sum_{b_{ii} \neq 0} \chi(b_{ii}) = 0, \text{ for } u+1 \leq i \leq n.$$

So we only need to consider the remaining two cases:  $u = n$  or  $\chi = 1$ .

Firstly, assume  $u = n$ . The sum (2.8) equals to

$$(2.10) \quad \sum_{(x_{ij}) \in \mathrm{GL}_n(k)} \frac{\chi(\det(x_{ij}))}{(q-1)^n q^{\binom{n}{2}}} \prod_{i=1}^n \sum_{b_{ii} \neq 0} \chi(b_{ii}) \lambda(b_{ii} x_{ii}) \cdot \prod_{i < j} \sum_{b_{ij} \in k} \lambda(b_{ij} x_{ji}).$$

For  $i < j$ ,

$$\sum_{b_{ij} \in k} \lambda(b_{ij} x_{ji}) \neq 0 \text{ if and only if } x_{ji} = 0.$$

Therefore, (2.10) equals to

$$(2.11) \quad \sum_{(x_{ij}) \in \mathrm{B}_n(k)} \frac{1}{(q-1)^n} \prod_{i=1}^n \sum_{b_{ii} \neq 0} \chi(b_{ii} x_{ii}) \lambda(b_{ii} x_{ii})$$

So we get

$$(2.12) \quad \sum_{X \in \mathrm{GL}_n(k)} \chi(\det X) \lambda(\mathrm{tr}_u X) = q^{\binom{n}{2}} G(\chi, \lambda)^n,$$

where

$$(2.13) \quad G(\chi, \lambda) = \sum_{x \in k^*} \chi(x) \lambda(x)$$

is the classical Gauss sum for  $k = \mathbb{F}_q$ .

Secondly, assume  $\chi$  is principal. The sum (2.8) equals to

$$(2.14) \quad \sum_{(x_{ij}) \in \mathrm{GL}_n(k)} \frac{1}{(q-1)^u q^{\binom{n}{2} - \binom{n-u}{2}}} \prod_{i=1}^u \sum_{b_{ii} \neq 0} \lambda(b_{ii} x_{ii}) \cdot \prod_{\substack{i < j \\ i \leq u}} \sum_{b_{ij} \in k} \lambda(b_{ij} x_{ji}).$$

The terms in the summation (2.14) over  $X = (x_{ij}) \in \mathrm{GL}_n(k)$  are nonzero if and only if

$$(2.15) \quad X = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \text{ for } A \in \mathrm{B}_u(k) \text{ and } D \in \mathrm{GL}_{n-u}(k).$$

In that case, they all equal to

$$\frac{(-1)^u}{(q-1)^u}.$$

Therefore,

$$(2.16) \quad \begin{aligned} & \sum_{X \in \mathrm{GL}_n(k)} \chi(\det X) \lambda(\mathrm{tr}_u X) \\ &= \frac{(-1)^u}{(q-1)^u} \# \{ (A, B, D) \mid A \in \mathrm{B}_u(k), B \in \mathrm{M}_{u \times (n-u)}(k), D \in \mathrm{GL}_{n-u}(k) \} \\ &= \frac{(-1)^u}{(q-1)^u} \left( (q-1)^u q^{\binom{n}{2} - \binom{n-u}{2}} \right) \cdot \prod_{i=0}^{n-u-1} (q^{n-u} - q^i) \\ &= q^{\binom{n}{2}} (-1)^u \prod_{i=1}^{n-u} (q^i - 1). \end{aligned}$$

Putting equations (2.5), (2.9), (2.12) and (2.16) all together, we get

**Theorem 2.1.** *Let  $u$  be the rank of  $U$  and  $\lambda$  be nontrivial. Then*

$$G_{\mathrm{GL}_n(k)}(U, \chi, \lambda) = \begin{cases} \bar{\chi}(\det U) q^{\binom{n}{2}} G(\chi, \lambda)^n & \text{if } u = n, \\ (-1)^u q^{\binom{n}{2}} \prod_{i=1}^{n-u} (q^i - 1) & \text{if } \chi = 1, \\ 0 & \text{if } u < n, \chi \neq 1, \end{cases}$$

where  $G(\chi, \lambda)$  is the classical Gauss sum defined in (2.13).

Letting  $u = 1$  in Theorem 2.1, we get the following corollary.

**Corollary 2.2.** *Uniformly over all nonzero matrices  $U \in M_n(\mathbb{F}_q)$  and non-trivial additive characters  $\lambda$ , we have*

$$G_{\mathrm{GL}_n(k)}(U, \lambda) = O(q^{n^2-n}),$$

where the implied constant in the symbol “ $O$ ” depends only on  $n$ .

**Remark 2.3.** Ferguson, Hoffman, Luca, Ostafe and Shparlinski in [2] obtained  $G_{\mathrm{GL}_n(k)}(U, \lambda) = O(q^{n^2-5/2})$ . (See Lemma 3 of [2]). Corollary 2.2 improves their bounds. Moreover, from our proof, it is easily seen that the bound  $O(q^{n^2-n})$  can not be improved. So Lemma 3 of [2] should be strengthened as

$$G_{\mathrm{GL}_n(k)}(U, \lambda) = O(q^{n^2-5/2}), \text{ for } n > 2.$$

**2.2. The case of  $\mathrm{SL}_n(k)$ .** If  $P, Q \in \mathrm{SL}_n(k)$ , the same argument as (2.2) shows that

$$(2.17) \quad G_{\mathrm{SL}_n(k)}(U, \lambda) = G_{\mathrm{SL}_n(k)}(PUQ, \lambda).$$

If  $u < n$ , from (2.4), we can assume  $P, Q \in \mathrm{SL}_n(k)$ , then

$$(2.18) \quad G_{\mathrm{SL}_n(k)}(U, \lambda) = \sum_{X \in \mathrm{SL}_n(k)} \lambda(\mathrm{tr}_u X).$$

Let  $D_h \in \mathrm{GL}_n(k)$  be the diagonal matrix  $\mathrm{Diag}(1, 1, \dots, 1, h)$ , where  $h \in k^*$ . Every element  $Y$  of  $\mathrm{GL}_n(k)$  can be uniquely written as  $Y = D_h X$  with  $X \in \mathrm{SL}_n(k)$  and  $h = \det X$ . So

$$\sum_{X \in \mathrm{SL}_n(k)} \lambda(\mathrm{tr}_u X) = \sum_{X \in \mathrm{SL}_n(k)} \frac{1}{q-1} \sum_{h \neq 0} \lambda(\mathrm{tr}_u D_h X) = \frac{1}{q-1} \sum_{Y \in \mathrm{GL}_n(k)} \lambda(\mathrm{tr}_u Y).$$

Therefore, from Theorem 2.1, we get

$$(2.19) \quad G_{\mathrm{SL}_n(k)}(U, \lambda) = \frac{1}{q-1} G_{\mathrm{GL}_n(k)}(U, 1, \lambda) = (-1)^u q^{\binom{n}{2}} \prod_{i=2}^{n-u} (q^i - 1).$$

Now we consider the case  $u = n$ . Let  $\tilde{\mathrm{B}}_n(k)$  be the Borel subgroup of  $\mathrm{SL}_n(k)$ , i.e., the group of upper triangular matrices with determinate 1.

Using the same method as in the case of  $\mathrm{GL}_n(k)$ , we get  
(2.20)

$$\begin{aligned}
& G_{\mathrm{SL}_n(k)}(U, \lambda) \\
&= \sum_{X \in \mathrm{SL}_n(k)} \lambda(\mathrm{tr} U^t X) \\
&= \sum_{\det X = \det U} \lambda(\mathrm{tr} X) \\
&= \frac{1}{(q-1)^{n-1} q^{\binom{n}{2}}} \sum_{B \in \tilde{\mathrm{B}}_n(k)} \sum_{\det X = \det U} \lambda(\mathrm{tr} BX) \\
&= \sum_{\det X = \det U} \frac{1}{(q-1)^{n-1} q^{\binom{n}{2}}} \sum_{B \in \tilde{\mathrm{B}}_n(k)} \lambda(\mathrm{tr} BX) \\
&= \sum_{\det(x_{ij}) = \det U} \frac{1}{(q-1)^{n-1} q^{\binom{n}{2}}} \sum_{(b_{ij}) \in \tilde{\mathrm{B}}_n(k)} \lambda\left(\sum_{i \leq j} b_{ij} x_{ji}\right) \\
&= \sum_{\det(x_{ij}) = \det U} \frac{1}{(q-1)^{n-1} q^{\binom{n}{2}}} \sum_{b_{11} \cdots b_{nn} = 1} \lambda\left(\sum_{i=1}^n b_{ii} x_{ii}\right) \cdot \prod_{i < j} \sum_{b_{ij} \in k} \lambda(b_{ij} x_{ji}) \\
&= \sum_{\substack{(x_{ij}) \in \mathrm{B}_n(k) \\ x_{11} \cdots x_{nn} = \det U}} \frac{1}{(q-1)^{n-1}} \sum_{b_{11} \cdots b_{nn} = 1} \lambda\left(\sum_{i=1}^n b_{ii} x_{ii}\right) \\
&= q^{\binom{n}{2}} K_n(\lambda, \det U),
\end{aligned}$$

where

$$(2.21) \quad K_n(\lambda, y) = \sum_{x_1 x_2 \cdots x_n = y} \lambda(x_1 + x_2 + \cdots + x_n), \text{ for } y \in k^*,$$

is the Kloosterman sum for  $k = \mathbb{F}_q$ .

Summing up, we get

**Theorem 2.4.** *Let  $u$  be the rank of  $U$  and  $\lambda$  be nontrivial. Then*

$$G_{\mathrm{SL}_n(k)}(U, \lambda) = \begin{cases} q^{\binom{n}{2}} K_n(\lambda, \det U) & \text{if } u = n, \\ (-1)^u q^{\binom{n}{2}} \prod_{i=2}^{n-u} (q^i - 1) & \text{if } u < n. \end{cases}$$

**Corollary 2.5.** *Uniformly over all nonzero matrices  $U \in M_n(\mathbb{F}_q)$  and non-trivial additive characters  $\lambda$ , we have*

$$G_{\mathrm{SL}_n(k)}(U, \lambda) = \begin{cases} O(1) & \text{if } n = 1, \\ O(q^{\frac{3}{2}}) & \text{if } n = 2, \\ O(q^{n^2-n-1}) & \text{if } n \geq 3. \end{cases}$$

where the implied constant in the symbol “ $O$ ” depends only on  $n$ .

*Proof.* The case:  $n = 1$  is trivial. So we assume  $n \geq 2$ . From Delinge’s bound of Kloosterman sum (see Example 2 in [20]), we get  $K_n(\lambda, y) = O(q^{\frac{n-1}{2}})$ ,

where  $y \in k^*$ . Therefore, by Theorem 2.4, we get

$$G_{\text{SL}_n(k)}(U, \lambda) = O(\max\{q^{\frac{n^2-1}{2}}, q^{n^2-n-1}\}).$$

Note that  $n^2 - n - 1 > (n^2 - 1)/2$  if and only if  $n \geq 3$ . So we get the desired bound.  $\square$

**Remark 2.6.** Ferguson, Hoffman, Luca, Ostafe and Shparlinski in [2] obtained  $G_{\text{SL}_n(k)}(U, \lambda) = O(q^{n^2-2})$ . (See Lemma 4 of [2]). Corollary 2.5 improves their bounds.

### 3. COUNTING INVERTIBLE MATRICES WITH GIVEN TRACE

For  $\beta \in k$ , let

$$(3.1) \quad N_\beta = \#\{X \in \text{GL}_n(k) \mid \text{tr} X = \beta\}.$$

The usual way of computing  $N_\beta$  involves the Bruhat decomposition of  $\text{GL}_n(k)$ , e.g. see [24, Prop. 1.10.15]. In this note, as an application of Theorem 2.1, we can calculate  $N_\beta$  purely by the method of exponential sums.

Since trace is a linear function, we get  $N_\beta = N_{h\beta}$ , for  $h, \beta \in k^*$ . So  $N_h = N_1$ , for  $h \in k^*$ . Then, we have

$$(3.2) \quad N_0 + (q - 1)N_1 = \#\text{GL}_n(k) = \prod_{i=0}^{n-1} (q^n - q^i)$$

and

$$(3.3) \quad \begin{aligned} G_{\text{GL}_n(k)}(I, \lambda) &= \sum_{X \in \text{GL}_n(k)} \lambda(\text{tr } X) \\ &= N_0 \lambda(0) + N_1 \sum_{h \in k^*} \lambda(h) = N_0 - N_1. \end{aligned}$$

Combining Theorem 2.1, equations (3.2) and (3.3), we get

**Theorem 3.1.** *Let  $h \in k^*$ . Then*

$$\begin{aligned} N_0 &= q^{\binom{n}{2}}(q - 1) \left( (-1)^n + \prod_{i=2}^n (q^i - 1) \right) / q, \\ N_h &= q^{\binom{n}{2}}(q - 1) \left( (-1)^{n-1} / (q - 1) + \prod_{i=2}^n (q^i - 1) \right) / q. \end{aligned}$$

**Acknowledgement** The first author is supported by the National Natural Science Foundation of China (Grant No. 11001145 and Grant No. 11071277). The second author is supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (2011-0001184).

## REFERENCES

- [1] M. Eichler, Allgemeine Kongruenz-Klasseneinteilungen der Ideale einfacher Algebren über algebraischen Zahlkörpern und ihre L-Reihen. J. Reine Angew. Math. 179 (1937), 227–251.
- [2] R. Ferguson, C. Hoffman, F. Luca, A. Ostafe, I. E. Shparlinski, Some additive combinatorics problems in matrix rings, Rev. Mat. Complut. (23) 2010, 501–513.
- [3] J. Fulman, A New Bound for Kloosterman Sums, arXiv:math/0105172v3.
- [4] S. Hu, Y. Li, On a uniformly distributed phenomenon in matrix groups, arXiv:1103.3928.
- [5] D. S. Kim, Gauss sums for classical groups over a finite field, Number theory, geometry and related topics (Iksan City, 1995), 97–101, Pyungsan Inst. Math. Sci., Seoul, 1996.
- [6] D. S. Kim, I.-S. Lee, Gauss sums for  $O^+(2n, q)$ , Acta Arith. 78 (1996), 75–89.
- [7] D. S. Kim, Gauss sums for  $O^-(2n, q)$ , Acta Arith. 80 (1997), 343–365.
- [8] D. S. Kim, Gauss sums for general and special linear groups over a finite field, Arch. Math. (Basel) 69 (1997), 297–304.
- [9] D. S. Kim, Y.-H. Park, Gauss sums for orthogonal groups over a finite field of characteristic two, Acta Arith. 82 (1997), 331–357.
- [10] D. S. Kim, Gauss sums for  $U(2n+1, q^2)$ , J. Korean Math. Soc. 34 (1997), 871–894.
- [11] D. S. Kim, Gauss sums for  $O(2n+1, q)$ , Finite Fields Appl. 4 (1998), 62–86.
- [12] D. S. Kim, Gauss sums for symplectic groups over a finite field, Monatsh. Math. 126 (1998), 55–71.
- [13] D. S. Kim, Exponential sums for symplectic groups and their applications, Acta Arith. 88 (1999), 155–171.
- [14] D. S. Kim, Hodges’ Kloosterman sums. Number theory and related topics (Seoul, 1998), 119–133, Yonsei Univ. Inst. Math. Sci., Seoul, 2000.
- [15] D. S. Kim, Exponential sums for  $O^-(2n, q)$  and their applications, Acta Arith. 97 (2001), 67–86.
- [16] D. S. Kim, Exponential sums for  $O(2n+1, q)$  and their applications, Glasg. Math. J. 43 (2001), 219–235.
- [17] D. S. Kim, Exponential sums for  $O^+(2n, q)$  and their applications, Acta Math. Hungar. 91 (2001), 79–97.
- [18] D. S. Kim, Sums for  $U(2n, q^2)$  and their applications, Acta Arith. 101 (2002), 339–363.
- [19] D. S. Kim, Codes associated with special linear groups and power moments of multi-dimensional Kloosterman sums, Ann. Mat. Pura Appl. (4) 190 (2011), 61–76.
- [20] E. Kowalski, Some aspects and applications of the Riemann hypothesis over finite fields, Milan J. of Mathematics, 78 (2010), 179–220.
- [21] T. Kondo, On Gaussian sums attached to the general linear groups over finite fields. J. Math. Soc. Japan 15 (1963), 244–255.
- [22] E. Lamprecht, Struktur und Relationen allgemeiner Gaußscher Summen in endlichen Ringen I, II. J. Reine Angew. Math. 197 (1957), 1–48.
- [23] A. N. Skorobogatov, Exponential sums, the geometry of hyperplane sections, and some Diophantine problems, Isr. J. Math. 80 (1992), 359–379.
- [24] R. P. Stanley, Enumerative Combinatorics, vol. I, second edition, available at <http://math.mit.edu/~rstan/ec/ec1>.

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